Tractability of Kőnig Edge Deletion Problems

Diptapriyo Majumdar[1] Rian Neogi[2] Venkatesh Raman[3]
S. Vaishali[4]

[1]Royal Holloway, University of London, Egham, United Kingdom

[2]University of Waterloo, Canada

[3]The Institute of Mathematical Sciences, HBNI, Chennai, India

[4]University of California, Santa Barbara, United States

March 29, 2021, BCTCS, Liverpool, United Kingdom

Outline

1. Kőnig graph and its properties
2. Parameterized Complexity
3. Kőnig Edge Deletion Problem
4. Related works
5. Conclusions

Kőnig graph

Kőnig graph

- Let $G = (V, E)$ be an undirected graph.

Kőnig graph

- Let $G = (V, E)$ be an undirected graph.
- A set of vertices $S \subseteq V(G)$ is said to be a *vertex cover* of G, if for every $uv \in E(G)$, $u \in S$, or $v \in S$.
- A set of edges $M \subseteq E(G)$ is said to be a *matching of* G if for no two edges of M share any endpoint.
- Minimum vertex cover size of G is $\mathsf{vc}(G)$, and maximum matching size of G is $\mu(G)$.

Kőnig graph

- Let $G = (V, E)$ be an undirected graph.
- A set of vertices $S \subseteq V(G)$ is said to be a *vertex cover* of G, if for every $uv \in E(G)$, $u \in S$, or $v \in S$.
- A set of edges $M \subseteq E(G)$ is said to be a *matching of* G if for no two edges of M share any endpoint.
- Minimum vertex cover size of G is $\mathsf{vc}(G)$, and maximum matching size of G is $\mu(G)$.
- For any graph $\mathsf{vc}(G) \geq \mu(G)$.

Kőnig graph

- Let $G = (V, E)$ be an undirected graph.
- A set of vertices $S \subseteq V(G)$ is said to be a *vertex cover* of G, if for every $uv \in E(G)$, $u \in S$, or $v \in S$.
- A set of edges $M \subseteq E(G)$ is said to be a *matching of* G if for no two edges of M share any endpoint.
- Minimum vertex cover size of G is $\mathsf{vc}(G)$, and maximum matching size of G is $\mu(G)$.
- For any graph $\mathsf{vc}(G) \geq \mu(G)$.
- A graph is said to be a *Kőnig graph* when $\mu(G) = \mathsf{vc}(G)$.

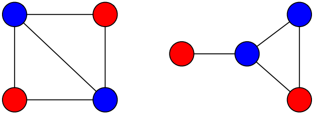
Bipartite and Kőnig graphs

Bipartite and Kőnig graphs

- A graph G is bipartite if $V(G) = A \uplus B$ and for every edge $uv \in E(G)$, $u \in A$ and $v \in B$.

Bipartite and Kőnig graphs

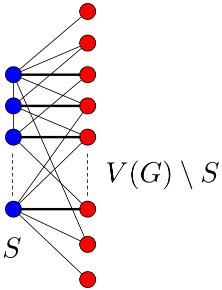
- A graph G is bipartite if $V(G) = A \uplus B$ and for every edge $uv \in E(G)$, $u \in A$ and $v \in B$.
- All bipartite graphs are Kőnig graphs, but not the converse.



Characteristics of Kőnig graph

Characteristics of Kőnig graph

- A graph G is a Kőnig graph if and only if for every minimum vertex cover S, there exists a matching M across $(S, V \setminus S)$, and saturating S.



Vertex Cover and Matching LP

- LP Relaxation for Vertex Cover:
  minimize $\sum_{v \in V(G)} x_v$ subject to
  for all $uv \in E(G), x_u + x_v \geq 1$ and
  for all $v \in V(G), 0 \leq x_v \leq 1$.

Vertex Cover and Matching LP

- LP Relaxation for Vertex Cover:
  minimize $\sum_{v \in V(G)} x_v$ subject to
  for all $uv \in E(G), x_u + x_v \geq 1$ and
  for all $v \in V(G), 0 \leq x_v \leq 1$.
- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.

Vertex Cover and Matching LP

- LP Relaxation for Vertex Cover:
  minimize $\sum_{v \in V(G)} x_v$ subject to
  for all $uv \in E(G), x_u + x_v \geq 1$ and
  for all $v \in V(G), 0 \leq x_v \leq 1$.
- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.
- $\mu(G) \leq \mu_f(G) = \mathsf{vc}_f(G) \leq \mathsf{vc}(G)$.

Vertex Cover and Matching LP

- LP Relaxation for Vertex Cover:
  minimize $\sum_{v \in V(G)} x_v$ subject to
  for all $uv \in E(G), x_u + x_v \geq 1$ and
  for all $v \in V(G), 0 \leq x_v \leq 1$.
- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.
- $\mu(G) \leq \mu_f(G) = \mathsf{vc}_f(G) \leq \mathsf{vc}(G)$.
- A graph G is a Kőnig graph if $\mathsf{vc}_f(G) = \mathsf{vc}(G)$ (we pass this observation that we were unable to find out in literature).

Outline

1. Kőnig graph and its properties
2. Parameterized Complexity
3. Kőnig Edge Deletion Problem
4. Related works
5. Conclusions

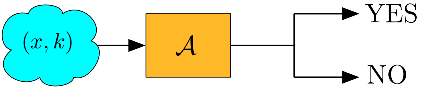
Fixed-Parameter Tractability

Fixed-Parameter Tractability

$$L \subseteq \Sigma^* \times \mathbb{N}$$

Fixed-Parameter Tractability

$$L \subseteq \Sigma^* \times \mathbb{N}$$

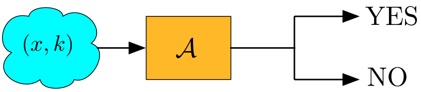


Yes if and only if $(x, k) \in L$

- $\mathcal{A}$ runs in $f(k)|x|^c$ time.

Fixed-Parameter Tractability

$$L \subseteq \Sigma^* \times \mathbb{N}$$

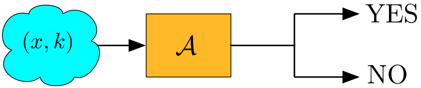


Yes if and only if $(x, k) \in L$

- $\mathcal{A}$ runs in $f(k)|x|^c$ time.
- $\mathcal{A}$ is called fixed-parameter algorithm (FPT algorithm).

Fixed-Parameter Tractability

$$L \subseteq \Sigma^* \times \mathbb{N}$$



Yes if and only if $(x, k) \in L$

- $\mathcal{A}$ runs in $f(k)|x|^c$ time.
- $\mathcal{A}$ is called fixed-parameter algorithm (FPT algorithm).
- Vertex Cover, $\{(G, k) | G$ has a vertex cover of size at most $k\}$.

Fixed-Parameter Intractability

- Some problems provably do not admit FPT algorithm.

Fixed-Parameter Intractability

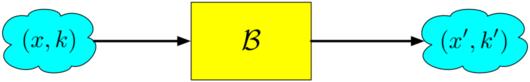
- Some problems provably do not admit FPT algorithm.
- Clique, Dominating Set.

Fixed-Parameter Intractability

- Some problems provably do not admit FPT algorithm.
- Clique, Dominating Set.
- FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \ldots$.

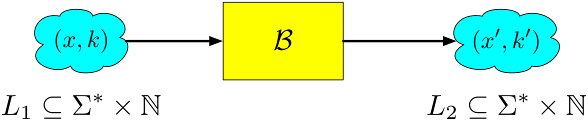
Parameterized Reduction

Parameterized Reduction



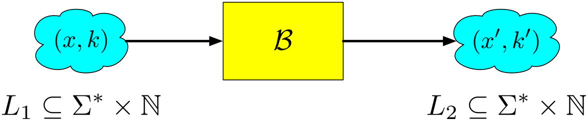
$L_1 \subseteq \Sigma^* \times \mathbb{N}$ $L_2 \subseteq \Sigma^* \times \mathbb{N}$

Parameterized Reduction



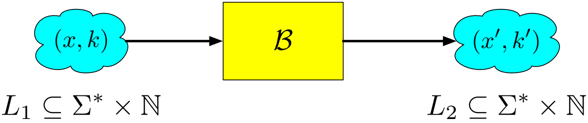
- $\mathcal{B}$ runs in $g(k)|x|^c$ time,

Parameterized Reduction



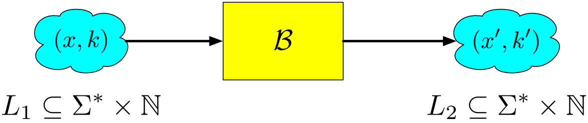
- $\mathcal{B}$ runs in $g(k)|x|^c$ time,
- $k' = f(k)$, and

Parameterized Reduction



- $\mathcal{B}$ runs in $g(k)|x|^c$ time,
- $k' = f(k)$, and
- $(x, k) \in L_1$ if and only if $(x', k') \in L_2$.

Parameterized Reduction



- $\mathcal{B}$ runs in $g(k)|x|^c$ time,
- $k' = f(k)$, and
- $(x, k) \in L_1$ if and only if $(x', k') \in L_2$.
- If for every problem $L' \in$ W[i], there is a parameterized reduction from L' to L, then L is W[i]-hard.

What is known?

What is known?

KONIG VERTEX DELETION
Input: An undirected graph $G = (V, E)$ and an integer k.
Parameter: k
Question: Is there $S \subseteq V(G)$ such that $|S| \leq k$, and $G - S$ is a Kőnig graph?

What is known?

KONIG VERTEX DELETION
Input: An undirected graph $G = (V, E)$ and an integer k.
Parameter: k
Question: Is there $S \subseteq V(G)$ such that $|S| \leq k$, and $G - S$ is a Kőnig graph?

- KONIG VERTEX DELETION is fixed-parameter tractable (FPT) [MRSS08,MRSSS11].

What is known?

KONIG VERTEX DELETION
Input: An undirected graph $G = (V, E)$ and an integer k.
Parameter: k
Question: Is there $S \subseteq V(G)$ such that $|S| \leq k$, and $G - S$ is a Kőnig graph?

- KONIG VERTEX DELETION is fixed-parameter tractable (FPT) [MRSS08,MRSSS11].
- ODD CYCLE TRANSVERSAL, EDGE BIPARTIZATION are some related problems that are also FPT.

Our Problems

Our Problems

Kőnig Edge Deletion
Input: An undirected graph $G = (V, E)$, and an integer k
Parameter: k
Question: Is there $F \subseteq E(G)$ such that $|F| \leq k$, and $G - F$ is a Kőnig graph?

Our Problems

Kőnig Edge Deletion
Input: An undirected graph $G = (V, E)$, and an integer k
Parameter: k
Question: Is there $F \subseteq E(G)$ such that $|F| \leq k$, and $G - F$ is a Kőnig graph?

- It was conjectured that Kőnig Edge Deletion is W[1]-hard [MRSS08, MRSSS11].

Our Problems

Kőnig Edge Deletion
Input: An undirected graph $G = (V, E)$, and an integer k
Parameter: k
Question: Is there $F \subseteq E(G)$ such that $|F| \leq k$, and $G - F$ is a Kőnig graph?

- It was conjectured that Kőnig Edge Deletion is W[1]-hard [MRSS08, MRSSS11].
- We settle this 11 years old conjecture by proving Kőnig Edge Deletion is W[1]-hard.

Our Problems

KŐNIG EDGE DELETION
Input: An undirected graph $G = (V, E)$, and an integer k
Parameter: k
Question: Is there $F \subseteq E(G)$ such that $|F| \leq k$, and $G - F$ is a Kőnig graph?

- It was conjectured that KŐNIG EDGE DELETION is W[1]-hard [MRSS08, MRSSS11].
- We settle this 11 years old conjecture by proving KŐNIG EDGE DELETION is W[1]-hard.
- In fact, our W-hardness result holds even when input graph has a perfect matching.

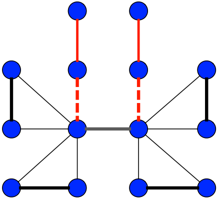
Our Problems

Our Problems

KED-Matching
Input: An undirected graph G, a maximum matching M, and an integer k.
Parameter: k
Question: Is there $F \subseteq E(G) \setminus M$ such that $|F| \leq k$, and $G - F$ is a Kőnig graph?

Our Problems

KED-Matching
Input: An undirected graph G, a maximum matching M, and an integer k.
Parameter: k
Question: Is there $F \subseteq E(G) \setminus M$ such that $|F| \leq k$, and $G - F$ is a Kőnig graph?



- We prove that KED-Matching is FPT.

Outline

1. Kőnig graph and its properties
2. Parameterized Complexity
3. Kőnig Edge Deletion Problem
4. Related works
5. Conclusions

Kőnig Edge Deletion

Kőnig Edge Deletion

- We provide a parameterized reduction from Independent Set to Kőnig Edge Deletion.

Kőnig Edge Deletion

- We provide a parameterized reduction from INDEPENDENT SET to KŐNIG EDGE DELETION.
- Let (G, k) be an instance of INDEPENDENT SET.

Kőnig Edge Deletion

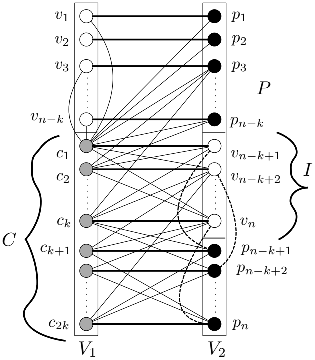
- We provide a parameterized reduction from INDEPENDENT SET to KŐNIG EDGE DELETION.
- Let (G, k) be an instance of INDEPENDENT SET.
- We assume without loss of generality that $k < n/2$, as otherwise $n \leq 2k$ and INDEPENDENT SET becomes FPT.

Kőnig Edge Deletion

- We provide a parameterized reduction from INDEPENDENT SET to KŐNIG EDGE DELETION.
- Let (G, k) be an instance of INDEPENDENT SET.
- We assume without loss of generality that $k < n/2$, as otherwise $n \leq 2k$ and INDEPENDENT SET becomes FPT.
- Let $V(G) = \{v_1, \ldots, v_n\}$.

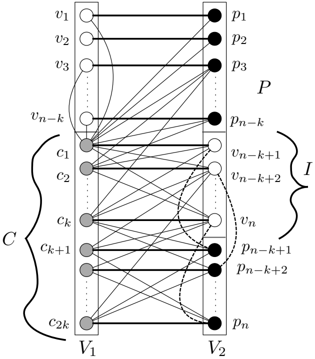
Kőnig Edge Deletion

Kőnig Edge Deletion



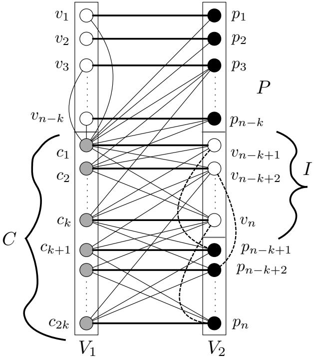
- (G', k') new instance, $V(G') = V(G) \cup C \cup P, k' = k$, and $E(G') = E(G) \cup \{v_i p_i | i \in [n]\} \cup \{cv, cp | c \in C, v \in V(G), p \in P\}$.

Kőnig Edge Deletion



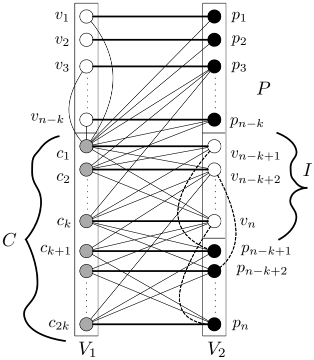
- (G', k') new instance, $V(G') = V(G) \cup C \cup P, k' = k$, and $E(G') = E(G) \cup \{v_i p_i | i \in [n]\} \cup \{cv, cp | c \in C, v \in V(G), p \in P\}$.
- $V_1 = (C \cup V(G)) \setminus I, V_2 = P \cup I$. We set $F = (V_2, V_2)$.

Kőnig Edge Deletion



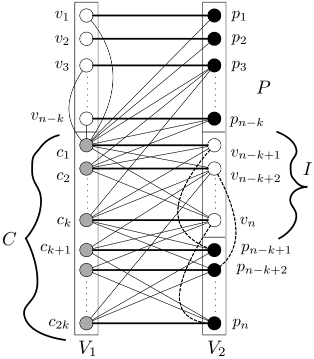
- (G', k') new instance, $V(G') = V(G) \cup C \cup P, k' = k$, and $E(G') = E(G) \cup \{v_i p_i | i \in [n]\} \cup \{cv, cp | c \in C, v \in V(G), p \in P\}$.
- $V_1 = (C \cup V(G)) \setminus I, V_2 = P \cup I$. We set $F = (V_2, V_2)$.
- $(\Rightarrow)$If $|I| = k$, then $|F| = k$ and F is a solution of (G', k').

Kőnig Edge Deletion



- (G', k') new instance, $V(G') = V(G) \cup C \cup P, k' = k$, and $E(G') = E(G) \cup \{v_i p_i | i \in [n]\} \cup \{cv, cp | c \in C, v \in V(G), p \in P\}$.

Kőnig Edge Deletion



- (G', k') new instance, $V(G') = V(G) \cup C \cup P, k' = k$, and $E(G') = E(G) \cup \{v_i p_i | i \in [n]\} \cup \{cv, cp | c \in C, v \in V(G), p \in P\}$.
- Let F be a Kőnig edge deletion set of G', and S be a minimum vertex cover of $G' - F$. Then, $|S| \leq n + k$.

Outline

1. Kőnig graph and its properties
2. Parameterized Complexity
3. Kőnig Edge Deletion Problem
4. Related works
5. Conclusions

Stable graph

- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.

Stable graph

- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.
- A graph G is a *stable graph* if $\mu_f(G) = \mu(G)$.

Stable graph

- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.
- A graph G is a *stable graph* if $\mu_f(G) = \mu(G)$.
- Stable Vertex Deletion: Is there a set at most k of vertices whose deletion results in a stable graph?

Stable graph

- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.
- A graph G is a *stable graph* if $\mu_f(G) = \mu(G)$.
- Stable Vertex Deletion: Is there a set at most k of vertices whose deletion results in a stable graph?
- (Weighted) Stable Vertex Deletion is polynomial time solvable [BCKPS2015].

Stable graph

- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.
- A graph G is a *stable graph* if $\mu_f(G) = \mu(G)$.
- Stable Vertex Deletion: Is there a set at most k of vertices whose deletion results in a stable graph?
- (Weighted) Stable Vertex Deletion is polynomial time solvable [BCKPS2015].
- Stable Edge Deletion is NP-Complete [BCKPS2015].

Stable graph

- LP Relaxation for Matching:
  maximize $\sum_{uv \in E(G)} y_{uv}$ subject to
  for all $v \in V(G)$, $\sum_{uv \in E(G)} y_{uv} \leq 1$, and
  for all $uv \in E(G)$, $0 \leq y_{uv} \leq 1$.
- A graph G is a *stable graph* if $\mu_f(G) = \mu(G)$.
- Stable Vertex Deletion: Is there a set at most k of vertices whose deletion results in a stable graph?
- (Weighted) Stable Vertex Deletion is polynomial time solvable [BCKPS2015].
- Stable Edge Deletion is NP-Complete [BCKPS2015].
- Edge Induced Stable Subgraph problem is FPT [MRSSS11,FTZFW18].

Outline

1. Kőnig graph and its properties
2. Parameterized Complexity
3. Kőnig Edge Deletion Problem
4. Related works
5. Conclusions

Conclusions and Open Problems

Conclusions and Open Problems

- What is the parameterized complexity status of STABLE EDGE DELETION problem?

THANK YOU